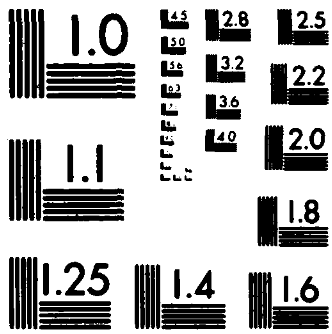
AD-A186 399 OVERESTIMATION OF ON-AXIS POWER DENSITY BY THE FRAUNHOFER ('FAR-FIELD') APPROXIMATION(U) MITRE CORP BEDFORD MA M M WEINER APR 87 MITRE-M87-37 1/1

UNCLASSIFIED ESD-TR-87-234 F19628-86-C-0001 F/G 20/3 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

DTIC FILE COPY

(12)

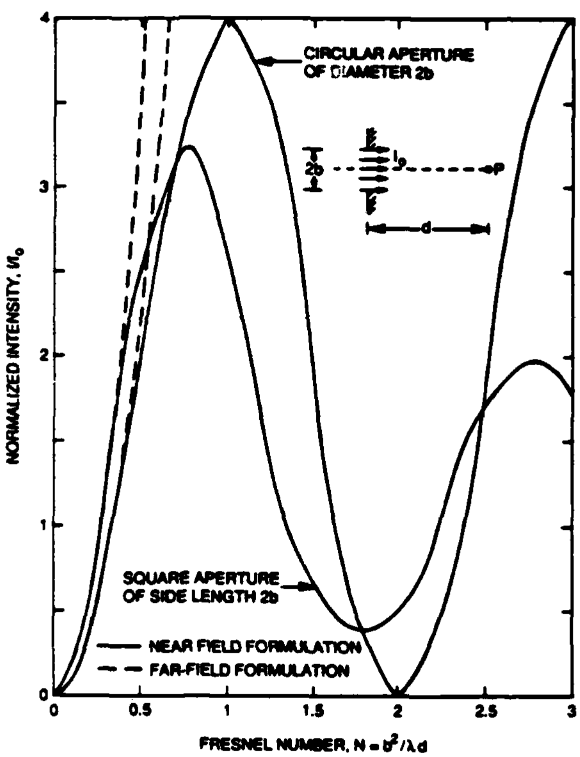
April 1987

M87-37

AD-A186 399

M. M. Weiner

Overestimation of On-Axis Power Density by the Fraunhofer ("Far-Field") Approximation



Approved for public release;
distribution unlimited.

MITRE
Bedford, Massachusetts

8c. ADDRESS (City, State, and ZIP Code)	Electronic Systems Division, AFSC Hanscom AFB, MA 01731-5000
10 SOURCE OF FUNDING NUMBERS	PROGRAM ELEMENT NO: 414J PROJECT NO: TASK NO: WORK UNIT ACCESSION NO:
11 TITLE (Include Security Classification)	Overestimation of On-Axis Power Density by the Fraunhofer ("Far-Field") Approximation
12 PERSONAL AUTHOR(S)	Weiner, Melvin M.
13a. TYPE OF REPORT	Final
13b TIME COVERED	FROM ______ TO ______
14 DATE OF REPORT (Year, Month, Day)	1987 April
15 PAGE COUNT	24
16 SUPPLEMENTARY NOTATION	

17 COSATI CODES		
FIELD	GROUP	SUB-GROUP

18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

Antenna Aperture
Aperture Distribution of Uniform Intensity and Phase
Circular Aperture (continued)

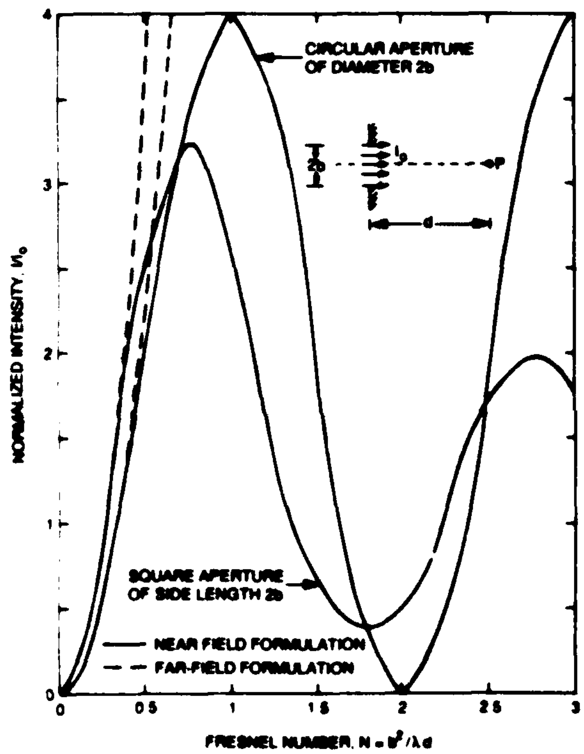
18. Constructive Interference
Destructive Interference
Diffraction
Far-Field
Fraunhofer
Fresnel
Fresnel-Kirchhoff Integral
Fresnel Number
Near-Field
On-Axis Intensity
On-Axis Power Density
Optical Aperture
Rectangular Aperture

April 1987

M87-37

M. M. Weiner

Overestimation of On-Axis Power Density by the Fraunhofer ("Far-Field") Approximation



CONTRACT SPONSOR ESD
CONTRACT NO. F19628-86-C-0001
PROJECT NO. 414J
DEPT. D86

MITRE

The MITRE Corporation
Bedford, Massachusetts

Department Approval: John F. Sullivan

J. F. Sullivan

MITRE Project Approval: R. B. Stevenson

R. B. Stevenson

ABSTRACT

The overestimation of power density by the Fraunhofer ("far-field") approximation to the more exact Fresnel ("near-field") diffraction integral is determined as a function of Fresnel number at arbitrary on-axis field points for rectangular and circular aperture illuminations of uniform amplitude and phase. For square and circular apertures, the overestimation is 2.8 percent and 1.3 percent, respectively, at a Fresnel number of 1/8 (corresponding to the far-field boundary distance $2D^2/\lambda$) and 518 percent and 147 percent, respectively, at a Fresnel number of 1.

*This work was sponsored by the Electronic Systems Division, Air Force Systems Command, Hanscom Air Force Base, Massachusetts.

TABLE OF CONTENTS

SECTION 1

INTRODUCTION

The Fraunhofer and Fresnel diffraction integrals are commonly used analytical tools for the evaluation of fields originating from illuminated apertures. These integrals are derived in appendix A from the Fresnel-Kirchhoff integral subject to small angle conditions. Whereas the exponential factor of the Fresnel integrand retains both linear and quadratic phase terms (as a function of the aperture coordinates) and is applicable to near-field points as well as far-field points, the Fraunhofer integrand retains only linear phase terms and is applicable only to far-field points. The Fraunhofer "far-field" formulation is, therefore, an approximation to the more exact Fresnel "near-field" formulation. (This statement is not applicable at a lens focal point where both linear and quadrature phase terms are zero.)

For on-axis field points and an aperture illumination of uniform phase, the Fraunhofer phase terms are zero, corresponding to totally constructive interference. However, for the same conditions, the Fresnel phase terms are non-zero, corresponding to destructive interference among interfering rays. Consequently, the far-field formulation yields a larger power density at an arbitrary on-axis field point than the more exact near-field formulation for an aperture illumination of uniform phase.

In this paper, the overestimation of power density by the far-field formulation is determined as a function of Fresnel number at arbitrary on-axis field points for rectangular and circular aperture illuminations of uniform amplitude and phase. For square and circular apertures, the overestimation is found to be 2.8 percent and 1.3 percent, respectively, at a Fresnel number of 1/8 (corresponding to the far-field boundary distance $2D^2/\lambda$) and 518 percent and 147 percent, respectively, at a Fresnel number of 1. The 1.3 percent result is compatible with a result reported earlier[1].

The Fresnel number is defined in section II. Far-field and near-field formulations are summarized in sections III and IV, respectively. A comparison of far-field and near-field formulations is given in section V.

SECTION 2

DEFINITION OF FRESNEL NUMBER

Consider a collimated beam, of uniform amplitude and phase with intensity I_o (W/m^2) at a wavelength λ(m), which is incident on a limiting aperture. For a rectangular aperture, the clear aperture is of width $2b_x$ and height $2b_y$ (see figure 1a). For a circular aperture, the clear aperture is of radius b (see figure 1b). In figure 1, a collimated beam corresponds to a source point S at a distance $d' = \infty$.

For a circular aperture of radius b, the number N of Fresnel zones, subtended by every point on the edge of the circular aperture at a field point P(0,0,d) on the optical axis at a distance d from the aperture, is defined as

$$N = \frac{(s+s') - (d+d')}{\lambda/2} \approx b^2/\lambda L;\ b \ll d,\ b \ll d' \qquad (1)$$

where

$$s = (b^2 + d^2)^{1/2}, \quad s' = (b^2 + d'^2)^{1/2} \quad \text{(see figure 1b)}$$

$$L = [(1/d) + (1/d')]^{-1}$$

For a collimated beam, $L = d$ and equation (1) reduces to

$$N \approx b^2/\lambda d;\ b \ll d,\ \text{collimated beam} \qquad (2)$$

For points P in the very near field, $N \rightarrow \infty$. For points P in the very far field, $N \rightarrow 0$. For a collimated beam, the boundary distance between the near field and far field is conventionally chosen as[1] $d = 2(2b)^2/\lambda$ which corresponds to a Fresnel number $N = b^2/\lambda d = 1/8$.

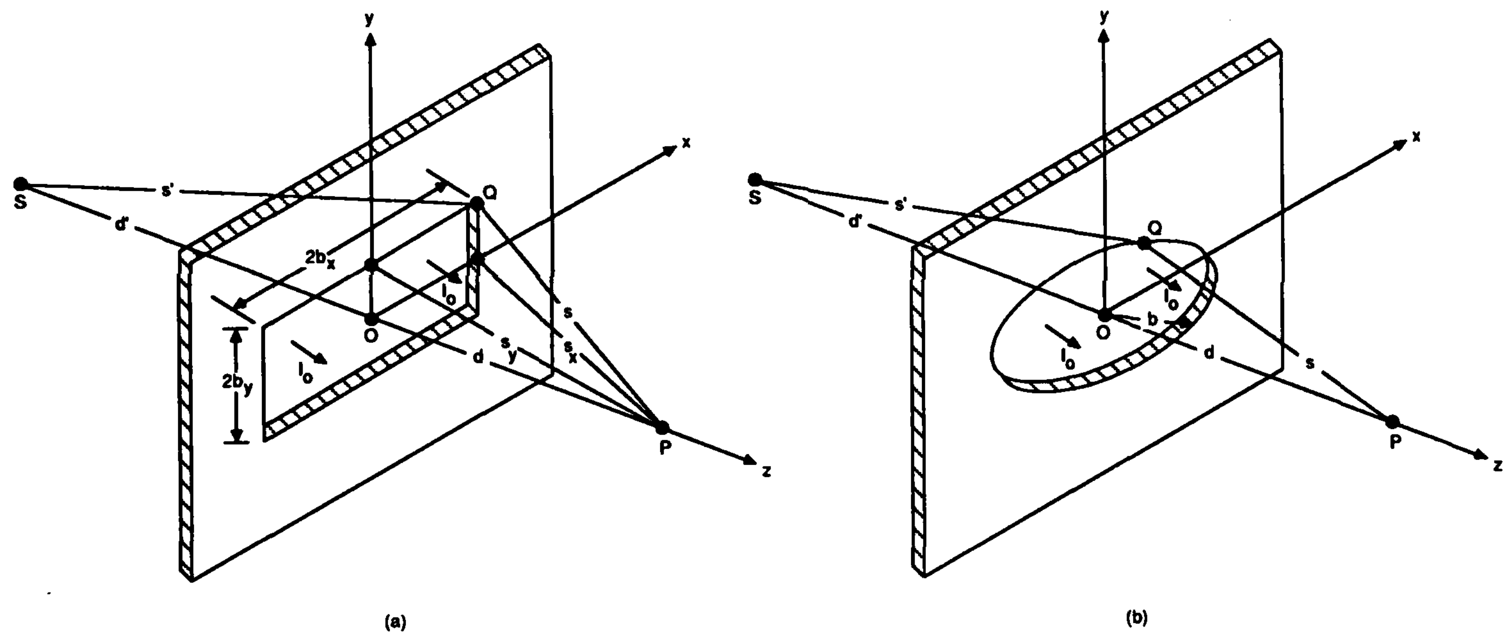


Figure 1. Limiting Aperture Geometry
(a) Rectangular Aperture
(b) Circular Aperture

Consider now the rectangular aperture of figure 1a. Not every point on the edge of the rectangular aperture subtends the same number of Fresnel zones (except in the very far field). Consequently, the effect of edge diffraction, on the intensity at a near-field point, is less pronounced for a rectangular or square aperture than for a circular aperture. Edge diffraction by a rectangular aperture is characterized by the Fresnel numbers, N_x and N_y, defined by

$$N_x = \frac{s_x - d}{\lambda/2} \approx b_x^2/(\lambda d);\ b_x \ll d,\ \text{collimated beam} \tag{3a}$$

$$N_y = \frac{s_y - d}{\lambda/2} \approx b_y^2/(\lambda d);\ b_y \ll d,\ \text{collimated beam} \tag{3b}$$

where

$$s_x = (b_x^2 + d^2)^{1/2}, \quad s_y = (b_y^2 + d^2)^{1/2} \quad \text{(see figure 1a)}$$

The Fresnel number N_x is the number of Fresnel zones subtended by every point on the edges $x = \pm b_x$ of the rectangular aperture at the field point P if the point source S were replaced by a line source of infinite extent parallel to the y axis and if the aperture were of infinite extent in the y direction. A similar statement applies to N_y if x and y are interchanged in the statement for N_x.

SECTION 3

FAR-FIELD FORMULATION

The on-axis intensity I (W/m^2) at a far-field point $P(x,y,z) = P(0,0,d)$ is found from the Fraunhofer diffraction integral of appendix A to be[2]

$$I(0,0,d) = |U(0,0,d)|^2 = \frac{I_o}{\lambda^2 d^2} \left| \iint_A dx_o dy_o \right|^2$$

$$= \frac{I_o A^2}{\lambda^2 d^2} = \frac{P_t A}{\lambda^2 d^2} = \frac{P_t g}{4\pi d^2} \text{ , collimated beam} \qquad (4)$$

where

$$A = \text{aperture area} = \begin{cases} 4b_x b_y & \text{, rectangular aperture} \\ \pi b^2 & \text{, circular aperture} \end{cases}$$

$$P_t = \text{ total power radiated} = I_o A$$

$$g = \text{ aperture (antenna) gain} = \frac{4\pi A}{\lambda^2}$$

$$= \begin{cases} \dfrac{4\pi}{(\lambda/2b_x)(\lambda/2b_y)} & \text{, rectangular aperture} \\ [\pi/(\lambda/2b)]^2 & \text{, circular aperture} \end{cases}$$

It will be noted from equation (4) that the beam is no longer collimated in the far field but appears to be spreading as though the aperture were a point source whose radiation is confined primarily to beam total angles of the order of $(\lambda/2b_x)$ in the x-z plane and $(\lambda/2b_y)$ in the y-z plane.

The intensity I at a field point P(0,0,d) can be expressed in terms of the number of Fresnel zones subtended by the aperture at the field point. Noting from equation (2) and equation (3) that $A = \pi N \lambda d$ for a circular aperture and that $A = 4\sqrt{N_x}\,\sqrt{N_y}\,\lambda d$ for a rectangular aperture, equation (4) reduces to

$$I = \begin{cases} I_o \, 16 \, N_x \, N_y \, , & \text{rectangular aperture} \quad (5a) \\ I_o \, \pi^2 \, N^2 \, , & \text{circular aperture} \quad (5b) \end{cases} ;$$

collimated beam, N_x, N_y, $N \lesssim 1/8$

It will be noted from equation (4) or equation (5) that in the far-field $I \to 0$ as $1/d^2 \to 0$ or equivalently $I \to 0$ as $N_x N_y$ (or N^2) $\to 0$.

SECTION 4

NEAR-FIELD FORMULATION

The intensity I (W/m^2) at any on-axis field point P(0,0,d) of a rectangular aperture with illumination of uniform amplitude and phase, is found from the Fresnel integral to be[3]

$$I = 4I_o \left[C^2(\sqrt{2N_x}) + S^2(\sqrt{2N_x})\right]\left[C^2(\sqrt{2N_y}) + S^2(\sqrt{2N_y})\right]; \quad (6)$$

collimated beam, $b_x \ll d$, $b_y \ll d$

where

$$C(w) = \int_0^w \cos\left[(\pi/2)t^2\right] dt = \text{Fresnel cosine integral}^{(4)}$$

$$S(w) = \int_0^w \sin\left[(\pi/2)t^2\right] dt = \text{Fresnel sine integral}^{(4)}$$

N_x, N_y = Fresnel numbers defined by equation (3)

I_o = aperture intensity (W/m^2)

The Fresnel integrals have the properties that[7]

$$S(w) = -S(-w), \quad C(w) = -C(-w) \quad (7a)$$

$$S(\infty) = C(\infty) = \tfrac{1}{2} \quad (7b)$$

$$S(0) = C(0) = 0 \quad (7c)$$

For $N_x = N_y = \infty$, equation (6) reduces to $I = I_o$ which is to be expected for points in the very near field. For $N_x = N_y = 0$, equation (6) reduces to $I = 0$ which is to be expected for points in the very far field.

Equation (6) reduces to equation (5a) in the limit of small Fresnel numbers. This result is demonstrated by noting that[5]

$$C(w) = w - \frac{(\pi/2)^2}{10} w^5 + \ldots. \tag{8a}$$

$$S(w) = \frac{\pi/2}{3} w^3 - \frac{(\pi/2)^3}{42} w^7 + \ldots. \tag{8b}$$

$$C^2(w) + S^2(w) = w^2 - \frac{4}{45} (\pi/2)^2 w^6 + \ldots. \tag{8c}$$

$$C^2(\sqrt{2N_x}) + S^2(\sqrt{2N_x}) = 2N_x - \frac{4}{45} (\pi/2)^2 (2N_x)^3 + \ldots. \tag{9a}$$

$$C^2(\sqrt{2N_y}) + S^2(\sqrt{2N_y}) = 2N_y - \frac{4}{45} (\pi/2)^2 (2N_y)^3 + \ldots. \tag{9b}$$

Substituting equations (9a) and (9b) with the condition $(4\pi^2/45)(N_x^2 + N_y^2) \ll 1$, into equation (6), $I \approx I_o\, 16\, N_x\, N_y$ which is the result given by equation 5(a).

For a circular aperture which subtends a small angle at a field point $P(0,0,d)$, the on-axis intensity I is found from the Fresnel integral to be

$$I = \left| \frac{-2\pi i}{\lambda} \sqrt{I_o} \frac{-\exp(i2\pi d/\lambda)}{d} \int_0^b \exp(i\pi r^2/\lambda d)\; r\, dr \right|^2$$

$$= I_o \left| 1 - \exp(-i\pi N) \right|^2 = I_o \left\{ [1 - \cos(\pi N)]^2 + \sin^2(\pi N) \right\}$$

$$= 4I_o \sin^2(\pi N/2); \quad \text{collimated beam, } b \ll d \qquad (10)$$

where

$$r = (x^2 + y^2)^{1/2}$$

$$N = b^2/\lambda d.$$

For $N = 1, 3, 5, \ldots.,$ $I = 4I_o$. For $N = 2, 4, 6, \ldots.,$ $I = 0$. For $N = \infty$, $I = I_o$. [Equation (10) does not extrapolate to this result because equation (10) is not valid for apertures which subtend a large angle at the field point. See reference [6] for circular apertures which subtend a large angle at the field point.] With the substitution $\exp(-i\pi N) \approx 1 - i\pi N$ for $(\pi^2 N^2/12) \ll 1$, equation (10) reduces to $I \approx I_o \pi^2 N^2$ which is compatible with the far-field result given by equation 5(b).

SECTION 5

COMPARISON OF FAR-FIELD AND NEAR-FIELD FORMULATIONS

The on-axis near-field intensities for square apertures of side length $2b_x = 2b_y = 2b$ and circular apertures of diameter $D = 2b$ with illumination of uniform amplitude and phase, are compared in figure 2 as a function of the Fresnel number $N_x = N_y = N = b^2/\lambda d$. The near-field formulations given by equations (6) and (10) are also compared with those obtained from the far-field formulations given by equations (5a) and (5b).

The peak intensities attained are $4I_o$ at Fresnel numbers N=1, 3, 5, ... for circular apertures and 3.24 I_o at a Fresnel number N = 3/4 for square apertures. The minimum intensities attained in the near field are 0 I_o at Fresnel numbers N = 2, 4, 6 ... for circular apertures and 0.37 I_o at a Fresnel number N = 7/4 for square apertures. The intensities obtained from the far-field formulations exceed those obtained from the more exact near-field formulations.

The ratios of these intensities are given in table 1 as a function of the Fresnel number N. For square and circular apertures, the far-field formulation overestimates the power density by 2.8 percent and 1.3 percent, respectively, at a Fresnel number of 1/8 (corresponding to the far-field boundary distance $2D^2/\lambda$) and 518 percent and 147 percent, respectively, at a Fresnel number of 1. The overestimation $[(\pi N/2)^2/\sin^2(\pi N/2)] - 1$ for circular apertures has been reported earlier[(1)]. Please note the round-off error in

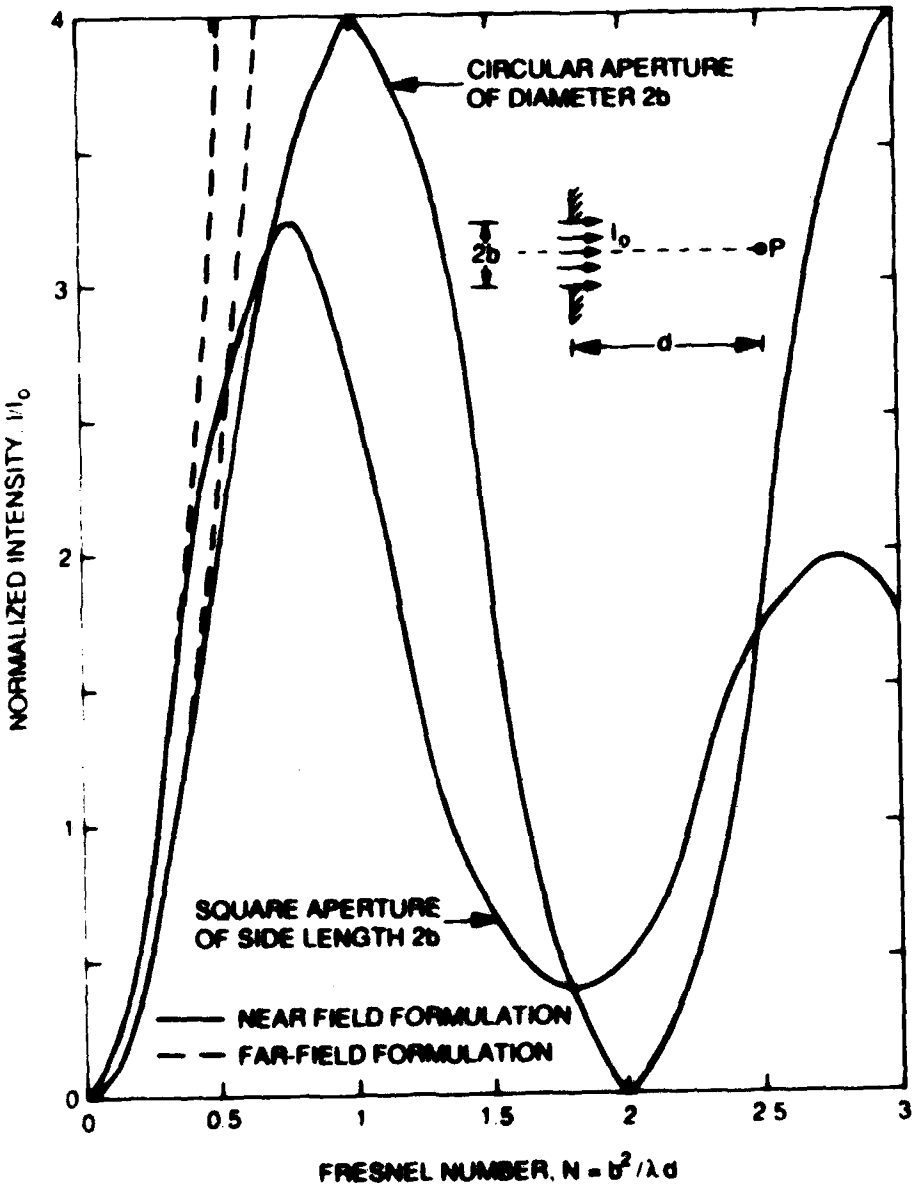


Figure 2. On-Axis Intensity for Square and Circular Apertures of Uniform Phase and Amplitude Illumination

Table 1. On-Axis Intensity from a Far-Field Formulation Normalized to that from a Near-Field Formulation

Fresnel Number, $N = b^2/\lambda d$	On-Axis Intensity Ratio, I_{far}/I_{near}: Square Aperture of Side Length 2b	On-Axis Intensity Ratio, I_{far}/I_{near}: Circular Aperture of Diameter 2b
0	1/1.00 = 1.000	1/1.00 = 1.000
0.125	0.25/0.2432 = 1.028	0.1542/0.1522 = 1.013
0.25	1/0.8954 = 1.117	0.6168/0.5858 = 1.053
0.50	4/2.5624 = 1.561	2.4674/2.0000 = 1.234
0.75	9/3.2428 = 2.775	5.5516/3.4142 = 1.626
1.00	16/2.5895 = 6.179	9.8696/4.0000 = 2.467
1.25	25/1.3020 = 19.201	15.4213/3.4142 = 4.517
1.50	36/0.6560 = 54.878	22.2066/2.0000 = 11.103
1.75	49/0.3684 = 133.008	30.2257/0.5858 = 51.597
2.00	64/0.5080 = 125.984	39.4784/0 = ∞
2.25	81/1.0244 = 79.071	47.9649/0.5858 = 85.293
2.50	100/1.7208 = 58.113	61.6850/2.0000 = 30.843
2.75	121/2.0064 = 60.307	74.6389/3.4142 = 21.861
3.00	144/1.7504 = 82.267	88.8264/4.0000 = 22.207
∞	∞/1.00 = ∞	∞/1.00 = ∞

reference [1] for a Fresnel number of 1/4 ($G/G_o \approx 0.95$ and not 0.94). For Fresnel numbers corresponding to constructive or weakly destructive interference, the overestimation is greater with square apertures than with circular apertures. However, for Fresnel numbers corresponding to totally or almost totally destructive interference, the overestimation is greater with circular apertures (see N=2 and N=2.25 in table 1). Constructive and destructive interference are more pronounced for circular apertures and, therefore, are in better and worse agreement, respectively, with the totally constructive on-axis interference of the Fraunhofer integral.

LIST OF REFERENCES

1. S. Silver, "Microwave Antenna Theory and Design" (McGraw-Hill, N.Y., 1949, MIT Reduction Lab Series, Vol. 12), p. 199.

2. M. Born and E. Wolf, "Principles of Optics" (Pergamon Press, Oxford, 1964), p. 386. Please note that equation (44) of reference 2 omits the factor $(1/d)^2$ [or equivalently, that the intensities in equation (44) are normalized to $(1/d)^2$].

3. G. R. Fowles, "Introduction to Modern Optics "(Holt, Rinehart and Winston, Inc., New York, 1968, p. 131. See also reference 2, pp. 428-433.

4. M. Abramovitz and I. Stegun, "Handbook of Mathematical Functions," U.S. Dept. of Commerce, National Bureau of Standards, Applied Mathematics Series 55, tenth printing, December 1972, p. 300, 7.3.1 and 7.3.2.

5. op. cit. 4, p. 301, 7.3.11 and 7.3.13.

6. op. cit. 2, pp. 370-375.

7. op. cit. 4, pp. 321-322

APPENDIX A

DERIVATION OF THE FRAUNHOFER AND FRESNEL DIFFRACTION INTEGRALS

For a source point $S(x',y',d')$, arbitrary aperture point $Q(x_o,y_o)$ with origin 0, and field point $P(x,y,d)$, the Fresnel-Kirchhoff diffraction integral for the amplitude $U(P)$ (proportional to either the scalar electric or magnetic field intensities) is given in the notation of figure 1 by[1]

$$U(P) = -\frac{Bi}{2\lambda}\iint_A \frac{\exp[ik(s'+s)]}{s's}[\cos(\vec{n},\vec{s}') - \cos(\vec{n},\vec{s})]dx_o dy_o \qquad \text{(A-1)}$$

where

$\vec{s}' = \vec{SQ}$, $s' = |\vec{SQ}|$

$\vec{s} = \vec{PQ}$, $s = |\vec{PQ}|$

$\vec{n}$ = unit vector normal to the aperture in the direction of the +z axis

A = area of the clear aperture

B = complex constant = $d'\sqrt{I_o}\, e^{-ikd'}$ which satisfies the boundary condition $U(0,0,0) = \sqrt{I_o}$.

(The aperture point $Q(x_o, y_o)$ is not restricted to the aperture point Q shown in figure 1).

For the small angle conditions

$$(\vec{n},\vec{s}') \approx (\vec{n}, \vec{SO}), \quad (\vec{n},\vec{s}) \approx (\vec{n},\vec{PO}) \qquad \text{(A-2)}$$

and the straight line condition

$$(\vec{SO}, \vec{PO}) = \pi \text{ radians} \qquad \text{(A-3)}$$

(1) M. Born and E. Wolf, "Principles of Optics "(Pergamon Press, Oxford 1964), pp. 382-383.

then

$$\cos(\vec{n},\vec{s}') \approx 1, \quad \cos(\vec{n},\vec{s}) \approx -1, \text{ [conditions (A-2) and (A-3)]} \qquad \text{(A-4)}$$

$$1/s's \approx 1/d'd \qquad \text{[condition (A-2)]} \qquad \text{(A-5)}$$

Expanding s' and s in a power series,

$$s' + s = d' + d - \frac{(xx_o + yy_o)}{d} + \frac{x_o^2 + y_o^2}{2L} - \frac{(xx_o + yy_o)^2}{2d^3} - \ldots \qquad \text{(A-6)}$$

$$\exp\,[ik(s'+s)] = \exp\,[ik(d'+d)]\,\exp\,(ikf) \qquad \text{(A-7)}$$

where

$$f \equiv -\frac{(xx_o + yy_o)}{d} + \frac{1}{2}\,\frac{x_o^2 + y_o^2}{L} - \frac{(xx_o + yy_o)^2}{2d^3} + \ldots .$$

$$L \equiv [(1/d) + (1/d')]^{-1}$$

Substituting equations (A-4) - (A-7) and the boundary condition $U(0,0,0) = \sqrt{I_o}$ into equation (A-11),

$$U(P) = \frac{-i\sqrt{I_o}}{\lambda d}\,\exp\,(ikd) \iint_A \exp(ikf)\,dx_o dy_o \qquad \text{(A-8)}$$

If only the first term in the above expansion for f is retained, then equation (A-8) reduces to

$$U(P) = \frac{-i\sqrt{I_o}}{\lambda d}\,\exp\,(ikd) \iint_A \exp\left[(-ik/d)\,(xx_o + yy_o)\right] dx_o dy_o \qquad \text{(A-9)}$$

Equation (A-9) is known as the Fraunhofer diffraction integral.

If only the first two terms in the above expansion for f are retained, then equation (A-8) reduces to

$$U(P) = \frac{-i\sqrt{I_o}}{\lambda d} \exp(ikd) \left(\iint_A \exp\left[(-ik/d)(xx_o + yy_o)\right] \exp\left[(ik/2L)(x_o^2 + y_o^2)\right] dx_o dy_o \right) \qquad \text{(A-10)}$$

Equation (A-10) is known as the Fresnel diffraction integral.

END
DATE
FILMED
DEC.
1987